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RANDOM INITIAL IMPERFECTIONS OF STRUCTURES

KIYOHIRO IKEDA
Department of Civil Engineering, Nagaoka University of Technology, Nagaoka, Niigata 940-21, Japan

and

KAZUO MUROTA
Department of Mathematical Engineering and Information Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

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Abstract—This paper is a theoretical study on random initial imperfections of structures. The explicit form of probability density function of the load-bearing capacity (critical load) of structures is derived for random initial imperfections based on a decomposition of the space of imperfection vectors into two orthogonal subspaces: the subspace that asymptotically affects the load-bearing capacity and the other that does not. Tight bounds on the range of load-bearing capacity are presented for various types of simple critical points. By means of the asymptotic theory of statistics, we show the inefficiency of a conventional random method that approximates the minimum load-bearing capacity by the minimum load for a number of random initial imperfections. The theoretical and empirical probability distribution functions for simple truss structures are compared to show the validity and effectiveness of the present method.

1. INTRODUCTION

The load-bearing capacity of structures has essential indeterministic nature due to the unavoidable presence of initial imperfections of structural members and materials. In order to deal with this indeterministic nature from a stochastic standpoint, it has been proposed to choose imperfection modes based on known probabilistic properties, typically the white noise (e.g., Elishakoff, 1988; Lindberg, 1988; Kirkpatrick and Holmes, 1989). This method of random initial imperfections, which obtains the load-bearing capacity (critical load) as a random variable, describes well the stochastic nature of initial imperfections. The minimum load-bearing capacity is to be approximated by the minimum load attained by a series of random initial imperfections. However, this method seems to have a major difficulty in practical applications in that large amounts of data have to be obtained by numerical analysis or experiment to estimate a lower bound and a distribution of load-bearing capacity. Nishino and Hartono (1989) analyzed 20,000 sets of initial imperfections to arrive at an estimate of the probability density function of load-bearing capacity of a reticulated truss dome.

Previous papers by the authors have introduced a method for determining the *critical initial imperfection* that asymptotically reduces the load-bearing capacity (critical load) most rapidly. This method affords the tight lower bound on the range of load-bearing capacity with a small amount of computation. The critical imperfection of various types of simple critical points has been determined in Ikeda and Murota (1990a,b), whereas Murota and Ikeda (1991) have dealt with double critical points, which appear generically in dome and shell structures with regular-polygonal symmetry.

Focusing on simple critical points, we investigate in this paper the probabilistic aspect of imperfections. Definite asymptotic lower and upper bounds of load-bearing capacity are presented by using the results of critical imperfection. We derive an explicit form of the probability density function of load-bearing capacity when the imperfections are chosen at random. This derivation is based on the fact that the space of imperfection vectors is to be divided into two orthogonal subspaces: the subspace that affects the load-bearing capacity and the other that does not. Since the former subspace is one-dimensional for simple critical points, this orthogonal decomposition can be easily computed. This makes it possible to derive various kinds of stochastic properties of load-bearing capacity by simple calculations.

This paper is organized as follows. In Section 2 a theory of random initial imperfections is introduced and the probability density function of load-bearing capacity is determined for various kinds of simple critical points. With the use of the asymptotic theory of statistics, we evaluate the accuracy of the random method to estimate the minimum load-bearing capacity. In Section 3 we demonstrate the probability density functions for simple example structures whose load-bearing capacities are governed by simple critical points, and show the validity and effectiveness of the present method.

2. THEORY

An asymptotic theory for random imperfections which is valid in the neighborhood of the critical point of a perfect system is presented in this section. We derive the probability density function of load-bearing capacity (critical load) of structures for various types of simple critical points. We also consider the distribution of the minimum load-bearing capacity to be observed in experiments, or for many random imperfections.

2.1. *Recapitulation*

The method proposed by Ikeda and Murota (1990a) for determining the critical imperfection at simple critical points of structures is summarized and extended in this subsection.

We consider a system of nonlinear equilibrium equations of a structure

$$\mathbf{H}(\lambda, \mathbf{u}, \mathbf{v}) = \mathbf{0},$$

where λ denotes a loading parameter; $\mathbf{u}$ indicates an N-dimensional nodal displacement (or position) vector; and $\mathbf{v}$ is a p-dimensional imperfection parameter vector. We assume $\mathbf{H}$ to be sufficiently smooth (or even analytic). We define $\mathbf{H}$ in such a manner that the eigenvalues of the tangent stiffness (Jacobian) matrix

$$J = J(\lambda, \mathbf{u}, \mathbf{v}) = (J_{ij}) = \left(\frac{\partial H_i}{\partial u_j}\right)$$

of $\mathbf{H}$ for the perfect system are all positive at $(\lambda, \mathbf{u}) = (0, \mathbf{0})$. This means that the system is originally (subcritically) in a stable state.

For a fixed $\mathbf{v}$, a set of solutions $(\lambda, \mathbf{u})$ of the above system of equations makes up equilibrium paths. Let $(\lambda_c, \mathbf{u}_c) = (\lambda_c(\mathbf{v}), \mathbf{u}_c(\mathbf{v}))$ denote the critical point of engineering interest, governing the load-bearing capacity of the structure described by $\mathbf{v}$. The tangent stiffness matrix is singular at $(\lambda_c, \mathbf{u}_c, \mathbf{v})$:

$$\det[J(\lambda_c, \mathbf{u}_c, \mathbf{v})] = 0.$$

In particular, this is satisfied by the critical point $(\lambda_c^0, \mathbf{u}_c^0, \mathbf{v}^0)$ of the perfect system, where superscript $(\cdot)^0$ refers to the perfect system.

We put

$$\lambda_c = \lambda_c^0 + \tilde{\lambda}_c,$$

where $\tilde{\lambda}_c$ means the increment of the critical load. To distinguish the mode and the magnitude of an imperfection, we write

$$\mathbf{v} = \mathbf{v}^0 + \varepsilon \mathbf{d}$$

with a scalar parameter $\varepsilon \geqslant 0$ and the imperfection mode vector $\mathbf{d}$ is normalized as

$$\mathbf{d}^T W \mathbf{d} = 1 \tag{1}$$

with respect to a positive definite matrix W (to be specified in accordance with the design principle).

The asymptotic behavior of the increment (increase or decrease) $\tilde{\lambda}_c$ of the critical load λ_c of the imperfect system is known to be expressed as

$$\tilde{\lambda}_c \sim C(\mathbf{d})\varepsilon^\rho \tag{2}$$

when ε is small. The increment $\tilde{\lambda}_c$ is characterized by ρ and $C(\mathbf{d})$. For a simple critical point, the coefficient $C(\mathbf{d})$ depends on $\mathbf{d}$ only through the variable

$$a = \xi^T B \mathbf{d}, \tag{3}$$

where $\xi = \xi_1$ is the critical eigenvector of

$$J^0 = J(\lambda_c^0, \mathbf{u}_c^0, \mathbf{v}^0)$$

satisfying

$$\xi^T J^0 = \mathbf{0}^T, \quad \|\xi\| = 1, \quad \xi^T \frac{\partial \mathbf{H}}{\partial \lambda} \geqslant 0$$

($\|\cdot\|$ denotes the Euclidean norm) and

$$B = (B_{ij}) = \left(\frac{\partial H_i}{\partial v_j}\bigg|_{(\lambda,\mathbf{u},\mathbf{v}) = (\lambda_c^0,\mathbf{u}_c^0,\mathbf{v}^0)} \bigg| i = 1,\ldots,N; j = 1,\ldots,p \right) \tag{4}$$

is an $N \times p$ constant matrix, called the *imperfection sensitivity matrix*.

The explicit form of ρ of eqn (2) has been obtained in Koiter (1945) and that of $C(\mathbf{d})$ in Ikeda and Murota (1990a) as follows:

$$\begin{cases} \rho = 1, & C(\mathbf{d}) = -C_0 a, & \text{at limit point;} \\ \rho = 1/2, & C(\mathbf{d}) = -C_0 \cdot |a|^{1/2}, & \text{at asymmetric point of bifurcation;} \\ \rho = 2/3, & C(\mathbf{d}) = -C_0 \cdot a^{2/3}, & \text{at unstable-symmetric point of bifurcation;} \\ \rho = 2/3, & C(\mathbf{d}) = C_0 \cdot a^{2/3}, & \text{at stable-symmetric point of bifurcation;} \end{cases} \tag{5}$$

where C_0 is a positive constant. The imperfection pattern vector $\mathbf{d}$ that maximizes $|C(\mathbf{d})|$ is called a *critical imperfection*.

Figure 1 shows general views of the perfect and imperfect equilibrium paths for these four types of critical points. The solid lines denote the equilibrium paths for the perfect

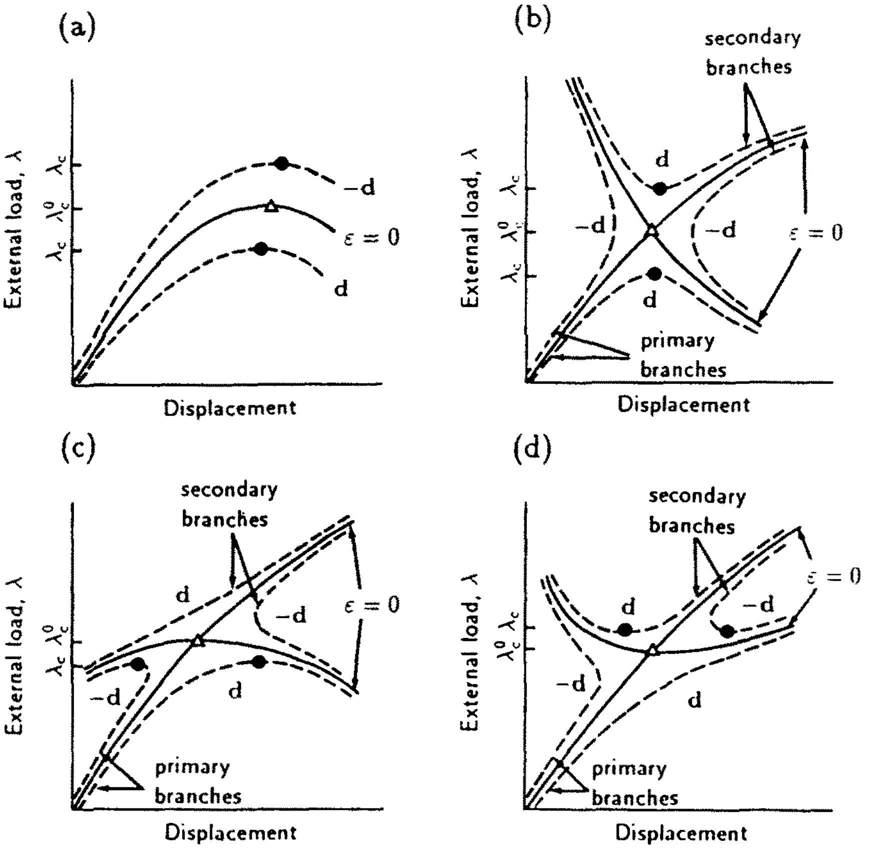

Fig. 1. Imperfections at various kinds of critical points. (a) Limit point. (b) Asymmetric point of bifurcation. (c) Unstable-symmetric point of bifurcation. (d) Stable-symmetric point of bifurcation. (△) Critical points for the perfect structure. (●) Critical points for the imperfect structure.

structure, whereas the dashed lines are those for imperfect ones. The open triangles (△) express the critical point on the path for the perfect system, the solid circles (●) the critical point for the imperfect system.

For a limit point the critical load $\tilde{\lambda}_c$ either decreases for **d** and increases for $-\mathbf{d}$, or increases for **d** and decreases for $-\mathbf{d}$. For a symmetric point of bifurcation **d** and $-\mathbf{d}$ exert the same influence on $\tilde{\lambda}_c$.

For an asymmetric point of bifurcation, **d** is to be divided into two categories according to the angle between **d** and a certain direction vector, say **k**. For **d** with $\mathbf{k}^T\mathbf{d} > 0$, a limit point exists on each of the primary and secondary branches of the imperfect system. Otherwise no critical (limit) point exists on the primary and secondary branches. Obviously, if **d** (respectively $-\mathbf{d}$) belongs to the first category, then $-\mathbf{d}$ (respectively **d**) belongs to the second. It should be noted that the critical load is reduced for **d** of the first category. On the other hand, for **d** of the second an external load λ increases stably over the critical load λ_c^0 for the perfect system and the critical load λ_c for the imperfect system cannot be defined. In this paper, we place an emphasis on **d** of the first category, which reduces the load-bearing capacity of the structure.

For a stable-symmetric point an external load λ can increase stably over the critical load λ_c^0 for the perfect structure. The critical load increment $\tilde{\lambda}_c$ in eqn (2) is associated with a pair of critical (limit) points on the secondary branches, which do not govern the load-bearing capacity of the structure. Accordingly, we will not be interested in the stable-symmetric point of bifurcation in the remainder of this paper.

For the first three types of critical points the values of critical load λ_c for the primary branches of the imperfect system can fall below the nominal value λ_c^0. Thus these types of points trigger instability by reducing the critical load, and hence are of practical importance.

The critical imperfection is determined as follows. Seeing that $C(\mathbf{d})$ is determined by $a = \xi^T B\mathbf{d}$, we decompose the space of imperfection parameters into the direct sum of the kernel of $\xi^T B$ and its orthogonal complement, where the orthogonality is defined with respect to W. Namely $\mathbf{d} = (d_1, \ldots, d_p)^T$ is decomposed as

$$\mathbf{d} = \mathbf{d}_{\text{eff}} + \mathbf{d}_{\text{ker}}, \tag{6}$$

where $\mathbf{d}_{\text{eff}}$ and $\mathbf{d}_{\text{ker}}$ are vectors satisfying

$$\xi^T B\mathbf{d}_{\text{ker}} = 0, \quad (\mathbf{d}_{\text{eff}})^T W\mathbf{d}_{\text{ker}} = 0.$$

Then we have

$$a = \xi^T B\mathbf{d} = \xi^T B\mathbf{d}_{\text{eff}}. \tag{7}$$

The kernel space is $(p-1)$-dimensional and the orthogonal complement is one-dimensional. It is easy to see that the orthogonal complement of $\ker(\xi^T B)$ is spanned by a vector

$$\mathbf{d}^* = W^{-1}B^T\xi/\alpha, \tag{8}$$

where the scaling factor

$$\alpha = (\xi^T BW^{-1}B^T\xi)^{1/2} \tag{9}$$

is introduced to make $\mathbf{d}^*$ satisfy

$$(\mathbf{d}^*)^T W\mathbf{d}^* = 1.$$

Hence we can define an orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_p)$ (with the orthogonality defined by W) of the space of $\mathbf{d}$ such that

$$\mathbf{e}_1 = \mathbf{d}^*;$$
$$\xi^T B\mathbf{e}_i = 0, \quad i = 2, \ldots, p.$$

Note that $(\mathbf{e}_2, \ldots, \mathbf{e}_p)$ spans $\ker(\xi^T B)$ and $\mathbf{e}_1 = \mathbf{d}^*$ its orthogonal complement. In terms of

$$\tilde{\mathbf{d}} \equiv (\tilde{d}_1, \ldots, \tilde{d}_p)^T = U\mathbf{d},$$

where $U^T = (\mathbf{e}_1, \ldots, \mathbf{e}_p)$ is the $p \times p$ orthogonal matrix representing the new basis, we can express $\mathbf{d}_{\text{eff}}$ and $\mathbf{d}_{\text{ker}}$ in eqn (6) as

$$\mathbf{d}_{\text{eff}} = \tilde{d}_1\mathbf{d}^* = U^T(\tilde{d}_1, 0, \ldots, 0)^T,$$
$$\mathbf{d}_{\text{ker}} = \sum_{i=2}^{p} \tilde{d}_i\mathbf{e}_i = U^T(0, \tilde{d}_2, \ldots, \tilde{d}_p)^T, \tag{10}$$

where $\tilde{d}_i = (\mathbf{e}_i)^T W\mathbf{d}$. Since $\mathbf{d}$ is normalized as eqn (1), we have $|\tilde{d}_i| \leqslant 1$ $(i = 1, \ldots, p)$.

Substitution of eqn (10) into eqn (7) results in

$$a = \alpha\tilde{d}_1. \tag{11}$$

Only the first component $\tilde{d}_1$ of $\tilde{\mathbf{d}}$ contributes to a, whereas $\tilde{d}_i$ $(i = 2, \ldots, p)$ have no influence on a. Thus the original minimization (or maximization) problem of $C(\mathbf{d})$ with p variables reduces to that of a of eqn (11) with one variable, which is much easier to handle. The above argument leads to the following important theorems.

Theorem 1. *For a given mode of imperfection* $\mathbf{d}$, *only the component* $\mathbf{d}_{\text{eff}} = \tilde{d}_1 \mathbf{d}^*$ *contributes to the change* $\tilde{\lambda}_c$ *of the critical load* λ_c *in the asymptotic sense, whereas* $\mathbf{d}_{\text{ker}}$ *has no effect on* $\tilde{\lambda}_c$. *Asymptotically the imperfections* $\mathbf{d}$ *and* $\mathbf{d}_{\text{eff}}$ *reduce the load-bearing capacity* λ_c *by the same amount.* □

Theorem 2. *When* $\mathbf{d}$ *changes under the constraint* (1), $|C(\mathbf{d})|$ *and hence* $|\tilde{\lambda}_c|$ *are maximized by* $\mathbf{d} = \mathbf{d}^*$, *i.e., by* $\tilde{\mathbf{d}} = (1, 0, \ldots, 0)^T$, *asymptotically as* $\varepsilon \to 0$. □

Theorem 3. *Under the constraint* (1), $|C(\mathbf{d})|$ *and hence* $|\tilde{\lambda}_c|$ *are minimized (i.e., nullified) by* $\mathbf{d} = \Sigma_{i=2}^{p} \tilde{d}_i \mathbf{e}_i$, *i.e., by* $\tilde{\mathbf{d}} = (0, \tilde{d}_2, \ldots, \tilde{d}_p)^T$, *asymptotically as* $\varepsilon \to 0$. □

We define the maximum and minimum of $C(\mathbf{d})$ with respect to $\mathbf{d}$ as

$$C_{\max} = \max\{C(\mathbf{d}) | \mathbf{d}^T W \mathbf{d} = 1\},$$
$$C_{\min} = \min\{C(\mathbf{d}) | \mathbf{d}^T W \mathbf{d} = 1\}.$$

We also define $\mathbf{d}_{\max}$ (respectively $\mathbf{d}_{\min}$) as one of the imperfections $\mathbf{d}$ that maximizes (respectively minimizes) $C(\mathbf{d})$. Table 1 shows the values of these variables for various kinds of simple critical points. Thus the values of $C(\mathbf{d})$, and hence those of $\tilde{\lambda}_c$, are bounded from above and below. For an asymmetric point of bifurcation, $\mathbf{d}_{\min}$ is equal to either $\mathbf{d}^*$ or $-\mathbf{d}^*$ and we need to actually compute the imperfect paths for $\pm\mathbf{d}^*$ to determine the one which yields the critical load.

Theorem 4. *Under the constraint* (1), $\tilde{\lambda}_c$ *stays in a bounded range*

$$C_{\min}\varepsilon^{\rho} \leqslant \tilde{\lambda}_c \leqslant C_{\max}\varepsilon^{\rho}$$

asymptotically as $\varepsilon \to 0$. □

2.2. *Random imperfection at simple points*

We are interested in the stochastic properties of the change $\tilde{\lambda}_c$ of critical load when imperfection mode $\mathbf{d}$ is given randomly.

If the weight matrix W is equal to the unit matrix I_p, the constraint (1) becomes

$$\|\mathbf{d}\| = 1, \tag{12}$$

which dictates that the imperfection mode $\mathbf{d}$ should stay on the unit sphere in $\mathbf{R}^p$. Then it would be natural to say that the imperfection mode $\mathbf{d}$ is random if it distributes uniformly on this unit sphere.

For a general W we decompose

$$W = V^T V$$

and define a transformation

$$\hat{\mathbf{d}} = V\mathbf{d}. \tag{13}$$

For this new variable $\hat{\mathbf{d}}$, the constraint (1) reduces to

Table 1. Values of $(C_{\min}, \mathbf{d}_{\min})$ and $(C_{\max}, \mathbf{d}_{\max})$ for a simple point

		Bifurcation	
Type of point	Limit point	Asymmetric	Unstable-symmetric
$C_{\min}$	$-C_0\alpha$	$-C_0\alpha^{1/2}$	$-C_0\alpha^{2/3}$
$\mathbf{d}_{\min}$	$\mathbf{d}^*$	$\mathbf{d}^*$ or $-\mathbf{d}^*$	$\mathbf{d}^*$ and $-\mathbf{d}^*$
$C_{\max}$	$C_0\alpha$	0	0
$\mathbf{d}_{\max}$	$-\mathbf{d}^*$	$\Sigma_{i=2}^{p}\tilde{d}_i\mathbf{e}_i$	$\Sigma_{i=2}^{p}\tilde{d}_i\mathbf{e}_i$

$$\|\tilde{\mathbf{d}}\| = 1, \tag{14}$$

which expresses the unit sphere in the space of $\tilde{\mathbf{d}}$. Hence for a general W, we will say that $\mathbf{d}$ is random under the constraint (1) if $\tilde{\mathbf{d}}$ distributes uniformly on this unit sphere. Note that this definition does not depend on the choice of V.

The following lemmas can be shown by elementary calculus.

Lemma 1. When $\tilde{\mathbf{d}} = (\tilde{d}_1, \ldots, \tilde{d}_p)^T$ distributes uniformly on the p-dimensional unit sphere (14), the joint probability density function of $(\tilde{d}_1, \ldots, \tilde{d}_q)$ (where $1 \leqslant q \leqslant p-1$) is given as

$$f(\tilde{d}_1, \ldots, \tilde{d}_q) = C_{qp}\left[1 - \sum_{i=1}^{q} (\tilde{d}_i)^2\right]^{(p-q-2)/2},$$

where

$$C_{qp} = \left\{\int \cdots \int_{\tilde{d}_1^2 + \cdots + \tilde{d}_q^2 < 1} \left[1 - \sum_{i=1}^{q} (\tilde{d}_i)^2\right]^{(p-q-2)/2} \mathrm{d}(\tilde{d}_1) \ldots \mathrm{d}(\tilde{d}_q)\right\}^{-1}$$

$$= \frac{\Gamma\left(\frac{p-q}{2}\right)(2\pi)^{q/2}}{\Gamma(p/2)}. \qquad \square$$

It is to be noted that a general case for which $\tilde{\mathbf{d}}$ does not distribute uniformly can be treated similarly.

Using the transformation (13), a of eqn (3) can be written as

$$a = \xi^T B\mathbf{d} = \xi^T B V^{-1}\tilde{\mathbf{d}}.$$

This shows that $a/\|\xi^T B V^{-1}\|$ is a one-dimensional projection of $\tilde{\mathbf{d}}$. On setting $q = 1$ in Lemma 1, we obtain the following by simple calculations.

Lemma 2. If $\mathbf{d}$ is a p-dimensional random vector with the constraint (14) (where $p \geqslant 2$), the probability density function of $a = \xi^T B\mathbf{d}$ is given by

$$f(a) = \frac{C_{1p}}{\alpha}\left[1 - \left(\frac{a}{\alpha}\right)^2\right]^{(p-3)/2}, \quad -\alpha < a < \alpha, \tag{15}$$

where α is defined by eqn (9); and

$$C_{1p} = \frac{2^{-p+2}}{B\left(\frac{p-1}{2}, \frac{p-1}{2}\right)} = \frac{(p-2)!!}{c\cdot(p-3)!!}, \quad c = \begin{cases} 2 & \text{if } p = \text{odd} \\ \pi & \text{if } p = \text{even} \end{cases},$$

and in which $B(\cdot,\cdot)$ is the beta function. $\square$

As a variable to measure the effect of random imperfections, we define a *normalized critical load increment*

$$\zeta = \frac{\tilde{\lambda}_c}{|C|_{\max}\varepsilon^{\rho}},$$

where $|C|_{\max}$ is the maximum of $|C(\mathbf{d})|$, which is equal to the greater value of $|C_{\max}|$ and $|C_{\min}|$. Note that $\zeta = 0$ corresponds to the nominal value, that is, the critical load λ_c^0 for the perfect system. Referring to eqns (2) and (5) and Table 1, we see that ζ asymptotically stays in the range:

$$\begin{cases} -1 \leqslant \zeta \leqslant 1, & \text{at limit point;} \\ -1 \leqslant \zeta \leqslant 0, & \text{at asymmetric point of bifurcation;} \\ -1 \leqslant \zeta \leqslant 0, & \text{at unstable-symmetric point of bifurcation.} \end{cases}$$

Note that the asymptotic range of ζ for an asymmetric point has been defined for $\mathbf{d}$ on the condition that a limit point exists on the primary branch of the imperfect system.

Using the probability density function (15) for a of eqn (3), we can arrive at the following theorem which gives the probability density function of the normalized critical load $\zeta = \tilde{\lambda}_c/(|C|_{\max}\varepsilon^{\rho})$ for random imperfections. Asymptotic expressions are derived with the use of Stirling's formula

$$\Gamma(x) \sim e^{-x}x^{x-(1/2)}\sqrt{2\pi} \quad (x \to +\infty).$$

Theorem 5. For a random imperfection mode $\mathbf{d}$ *with a constant* ε *and* $p \geqslant 2$, *the probability density function* $f(\zeta)$, *the expected value* $E[\zeta]$, *and the variance* Var $[\zeta]$ *of* ζ *are given for various kinds of critical points as follows.*

Limit point:

$$\begin{aligned} f(\zeta) &= C_{1p}(1-\zeta^2)^{(p-3)/2}, \quad -1 < \zeta < 1, \\ E[\zeta] &= 0, \\ E[|\zeta|] &= \frac{2C_{1p}}{p-1} \\ &\sim \sqrt{\frac{2}{\pi}}p^{-1/2} \quad (p \to +\infty), \\ \mathrm{Var}[\zeta] &= E[\zeta^2] = p^{-1}. \end{aligned}$$

Asymmetric point of bifurcation (conditional on the existence of a limit point):

$$\begin{aligned} f(\zeta) &= 4C_{1p}|\zeta|(1-\zeta^4)^{(p-3)/2}, \quad -1 < \zeta < 0, \\ E[\zeta] &= -C_{1p}B\left(\frac{3}{4}, \frac{p-1}{2}\right) \\ &\sim \frac{-2^{1/4}}{\sqrt{\pi}}\Gamma(\tfrac{3}{4})p^{-1/4} = -0.82218 \times p^{-1/4} \quad (p \to +\infty), \\ E[\zeta^2] &= \frac{2C_{1p}}{p-1} \\ &\sim \sqrt{\frac{2}{\pi}}p^{-1/2} \quad (p \to +\infty), \\ \mathrm{Var}[\zeta] &= \frac{2C_{1p}}{p-1} - \left[C_{1p}B\left(\frac{3}{4}, \frac{p-1}{2}\right)\right]^2 \\ &\sim \frac{\sqrt{2}}{\pi}\{\sqrt{\pi} - [\Gamma(\tfrac{3}{4})]^2\}p^{-1/2} = 0.12191 \times p^{-1/2} \quad (p \to +\infty). \end{aligned}$$

Unstable-symmetric point of bifurcation:

$$f(\zeta) = 3C_{1p}|\zeta|^{1/2}(1-|\zeta|^3)^{(p-3)/2}, \quad -1 < \zeta < 0,$$

$$E[\zeta] = -C_{1p}B\left(\frac{5}{6},\frac{p-1}{2}\right)$$

$$\sim \frac{-2^{1/3}}{\sqrt{\pi}}\Gamma(\tfrac{5}{6})p^{-1/3} = -0.80238 \times p^{-1/3} \quad (p \to +\infty),$$

$$E[\zeta^2] = C_{1p}B\left(\frac{7}{6},\frac{p-1}{2}\right)$$

$$\sim \frac{2^{2/3}}{\sqrt{\pi}}\Gamma(\tfrac{7}{6})p^{-2/3} = 0.83086 \times p^{-2/3} \quad (p \to +\infty),$$

$$\mathrm{Var}[\zeta] = C_{1p}B\left(\frac{7}{6},\frac{p-1}{2}\right) - \left[C_{1p}B\left(\frac{5}{6},\frac{p-1}{2}\right)\right]^2$$

$$\sim 0.18705 \times p^{-2/3} \quad (p \to +\infty).$$

□

It is to be noted that the probability distribution function of ζ for an asymmetric point has been defined as the conditional distribution given that a limit point exists on the primary branch of the imperfect system.

Figure 2 shows the probability density functions of these three types of critical points for $p = 2, 3, 4, 5, 12$ and 24. These distributions display strong dependency on the number p of imperfection parameters. The probability concentrates around $\zeta = 0$ as p increases. In fact, the expected value $E[\zeta]$ vanishes for a limit point and asymptotically vanishes as p tends to infinity for bifurcation points, and the standard deviation

$$\sqrt{\mathrm{Var}[\zeta]} \sim \begin{cases} p^{-1/2} & (p \to +\infty), \quad \text{at limit point;} \\ 0.34915 \times p^{-1/4} & (p \to +\infty), \quad \text{at symmetric point of bifurcation;} \\ 0.43249 \times p^{-1/3} & (p \to +\infty), \quad \text{at symmetric point of bifurcation} \end{cases}$$

asymptotically vanishes as $p \to \infty$.

This means that the minimum load ($\zeta = -1$) is not likely to be approximated by randomly chosen **d**, especially when p is large. In other words, the method of random imperfections will cease to be very effective for large p as a means of evaluating the minimum load. This indicates the importance of the theory of critical imperfection, which can evaluate the minimum load with a small amount of computation.

2.3. *Distribution of minimum value*

It would be natural and rational to approximate the minimum load-bearing capacity (critical load) by the minimum load achieved by a series of random imperfections **d**. In this

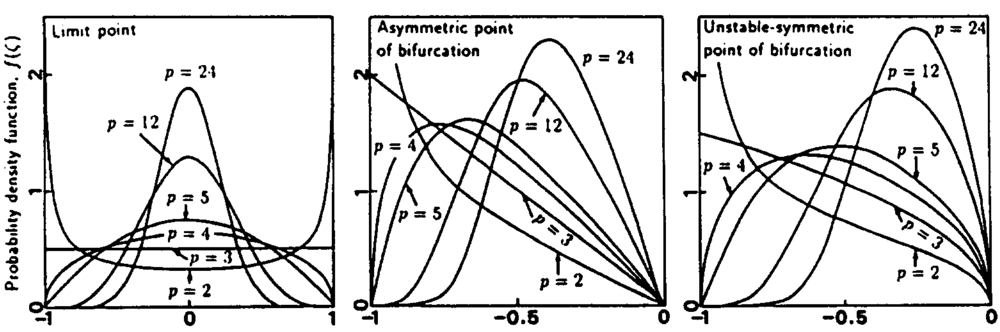

Fig. 2. Probability density function of normalized critical load increment ζ.

section we will evaluate the accuracy of this approximation. We will not consider here the stable-symmetric point of bifurcation, which does not govern the load-bearing capacity.

Let ζ_K be the minimum value of the normalized critical load ζ attained by K independent random imperfections. The cumulative distribution function $F_K(\zeta_K)$ of ζ_K is given by the following formula (see Kendall and Stuart, 1977, for the background for statistical theory):

$$F_K(\zeta_K) = 1-[1-F(\zeta_K)]^K,$$

where

$$F(\zeta) = \int_{\zeta_{\min}}^{\zeta} f(\zeta)\,\mathrm{d}\zeta$$

is the cumulative distribution function of ζ, and $\zeta_{\min} = -1$.

The asymptotic form of F_K (as $K \to +\infty$) is known as follows (see Theorem 2.1.5 of Galambos, 1978; also Kendall and Stuart, 1977).

Lemma 3. If there is a constant $\gamma > 0$ such that

$$\lim_{t \to +\infty} \frac{F(\zeta_{\min}+1/(tx))}{F(\zeta_{\min}+1/t)} = x^{-\gamma} \tag{16}$$

for all $x > 0$, then

$$\lim_{K \to +\infty} F_K(\zeta_{\min}+s_K x) = L_\gamma(x),$$

where

$$s_K = F^{-1}(1/K)-\zeta_{\min},$$

$$L_\gamma(x) = \begin{cases} 1-\exp(-x^\gamma) & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases} \qquad \square$$

From the concrete forms of $f(\zeta)$ given in Theorem 5 we see that (16) is satisfied with

$$\gamma = \frac{p-1}{2}$$

for all cases we consider here (i.e., limit point, asymmetric point of bifurcation, and unstable-symmetric point of bifurcation).

The scaling factor s_K represents the order of magnitude of the discrepancy between ζ_K and $\zeta_{\min}$. A simple calculation yields

$$s_K = O(K^{-1/\gamma}) = O(K^{-2/(p-1)})$$

as $K \to +\infty$. This shows

$$|\zeta_K-\zeta_{\min}| = O(K^{-2/(p-1)}) \tag{17}$$

as $K \to +\infty$, from which we see that the convergence of ζ_K to $\zeta_{\min}$ is extremely slow. This implies that, for realistic structures with large p, ζ_K will not be rapidly improved, as an approximation to the (normalized) minimum load-bearing capacity, with the increase of the number K of random imperfections.

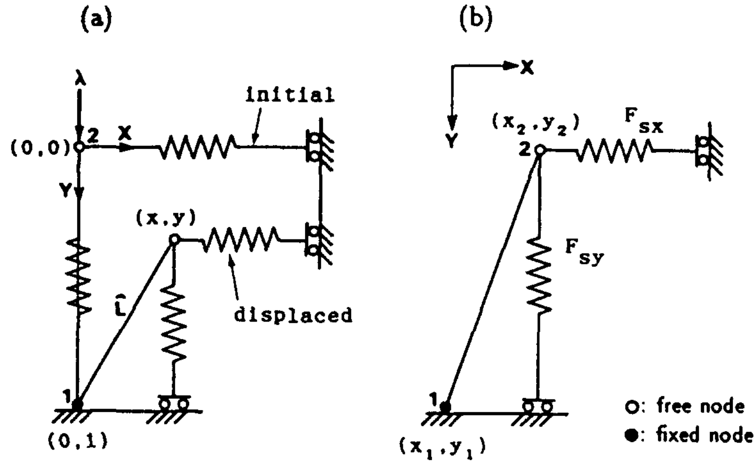

Fig. 3. Propped cantilever. (a) Perfect system. (b) Imperfect system.

3. NUMERICAL EXAMPLES

3.1. *Asymmetric point of bifurcation*

Consider the propped cantilever of Fig. 3 comprising a truss member, simply supported with a rigid foundation at node 1 and supported by horizontal and vertical springs at node 2. This example has also been used in Ikeda and Murota (1990a). A vertical load λ is applied to the free node 2. The set of equilibrium equations is

$$\mathbf{H}(\lambda, \mathbf{u}, \mathbf{v}) \equiv \begin{bmatrix} EA(1/L-1/\hat{L})(x-x_1)+F_{sx} \\ EA(1/L-1/\hat{L})(y-y_1)+F_{sy} \end{bmatrix} - \begin{pmatrix} 0 \\ \lambda \end{pmatrix} = \mathbf{0}, \tag{18}$$

where

$$L = [(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}; \quad \hat{L} = [(x-x_1)^2+(y-y_1)^2]^{1/2};$$

$$F_{sx} = EA\left[\frac{x-x_2}{L}+\left(\frac{x-x_2}{L}\right)^2\right]; \quad F_{sy} = EA\frac{y-y_2}{L};$$

$\mathbf{u} = (x, y)^T$ is the location of node 2 after displacement; (x_i, y_i) is the initial location of node i $(i = 1, 2)$; and F_{sx} and F_{sy} are the horizontal and vertical forces exerted by the springs, respectively. Note that the nonlinear spring F_{sx} is used to make the bifurcation point asymmetric.

The solid lines in Fig. 4 show equilibrium paths (λ versus x curves) computed for the

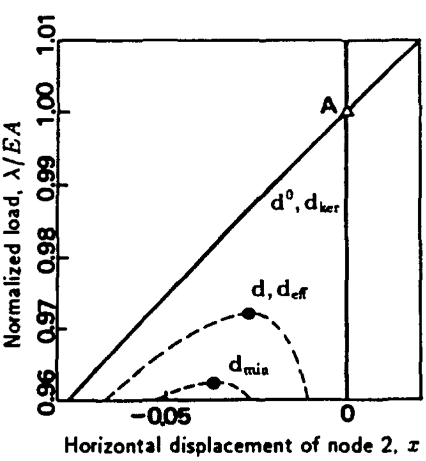

Fig. 4. Equilibrium paths (λ versus x curves) for the propped cantilever (asymmetric point of bifurcation). (△) Asymmetric points of bifurcation. (●) Limit points.

perfect cantilever from eqn (18). These paths consist of a main path and a pair of bifurcation paths branching at a simple asymmetric bifurcation point A. The critical eigenvector at A is $\xi = (1, 0)^T$ and $(\lambda_c^0, x_c^0, y_c^0) = (EA, 0, 1/2)$.

We choose (x_i, y_i) $(i = 1, 2)$ as imperfection parameters, and define

$$\mathbf{v} = (x_1, y_1, x_2, y_2)^T$$

as an imperfection parameter vector ($p = 4$). In the perfect case, we have

$$\mathbf{v}^0 = (0, 1, 0, 0)^T.$$

The weight matrix is chosen to be the unit matrix, that is,

$$W = I_4.$$

The imperfection sensitivity matrix

$$B = EA\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}$$

is obtained by differentiating $\mathbf{H}$ of (18) with respect to $\mathbf{v}$ and evaluating at $(\lambda, x, y, \mathbf{v}) = (\lambda_c^0, x_c^0, y_c^0, \mathbf{v}^0)$.

First we consider the imperfection pattern $\mathbf{d}_{\min}$, which should asymptotically yield the minimum of the critical load increment $\tilde{\lambda}_c$ among all imperfection modes with the same magnitude ε under constraint (1). The pattern $\mathbf{d}_{\min}$ is given as

$$\mathbf{d}_{\min} = \mathbf{d}^* = W^{-1}B^T\xi/\alpha = \frac{1}{\sqrt{2}}(1, 0, -1, 0)^T \tag{19}$$

by eqn (8) and Table 1. We have chosen $\mathbf{d}_{\min} = \mathbf{d}^*$ among $\pm\mathbf{d}^*$ since in this particular case the path for $\mathbf{d}^*$ has a limit point whereas the path for $-\mathbf{d}^*$ does not show any limit point.

Next we choose a random imperfection mode vector

$$\mathbf{d} = (0.5382, -0.4637, -0.2394, -0.6618)^T. \tag{20}$$

According to eqn (6) this random vector $\mathbf{d}$ is to be divided into two parts

$$\mathbf{d} = \mathbf{d}_{\text{eff}} + \mathbf{d}_{\text{ker}},$$

where

$$\mathbf{d}_{\text{eff}} = 0.3888 \times (1, 0, -1, 0)^T,$$
$$\mathbf{d}_{\text{ker}} = (0.1494, -0.4637, 0.1494, -0.6618)^T.$$

Figure 4 shows the equilibrium paths computed from eqn (18) for the imperfection pattern vectors $\mathbf{d}_{\min}$, $\mathbf{d}$, $\mathbf{d}_{\text{eff}}$ and $\mathbf{d}_{\text{ker}}$ with $\varepsilon = 10^{-3}$. The path for the perfect structure with $\mathbf{d}^0 = \mathbf{0}$ and that for the imperfect structure with $\mathbf{d}_{\text{ker}}$ are very close (but not identical) in values showing no discernible difference in the figure. Likewise the paths for $\mathbf{d}$ and $\mathbf{d}_{\text{eff}}$ are very close. It is to be noted that the imperfection $\mathbf{d}_{\text{ker}}$ displays the same bifurcation structure as the perfect structure, with an asymmetric point of bifurcation and a pair of branches.

The critical loads attained by $\mathbf{d}_{\min}$, $\mathbf{d}$, $\mathbf{d}_{\text{eff}}$ and $\mathbf{d}_{\text{ker}}$ are $0.963EA$, $0.972EA$, $0.972EA$ and $1.000EA$, respectively. Note that $\mathbf{d}_{\min}$ degrades the critical load most rapidly among these four imperfection vectors. The imperfections $\mathbf{d}$ and $\mathbf{d}_{\text{eff}}$ reduce the load-bearing capacity by approximately the same amount, in accordance with Theorem 1. The load-bearing capacity for $\mathbf{d}_{\text{ker}}$ is very close to the nominal critical load EA for the perfect case. This shows

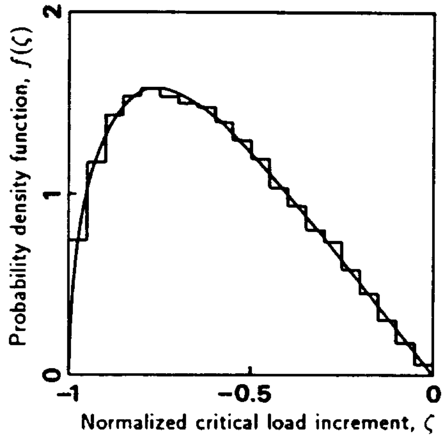

Fig. 5. Comparison of theoretical and empirical probability density function $f(\zeta)$ of normalized critical load increment ζ for the propped cantiliver (the empirical distribution is based on $K = 100{,}000$ samples).

the correctness of Theorem 3, which states that an imperfection vector belonging to the kernel space asymptotically exerts no influence on critical load. The critical load $0.972EA$ attained by the random mode $\mathbf{d}$ falls between the minimum load $0.963EA$ for $\mathbf{d}_{min}$ and the nominal load EA for the perfect case $\mathbf{d}^0$. This agrees with Theorem 4.

We have randomly chosen $K = 100{,}000$ imperfection modes, which yields a limit point, and computed the normalized load-bearing capacity (critical load) ζ for constant imperfection magnitude $\varepsilon = 10^{-3}$. Figure 5 shows a comparison of the histogram obtained from these 100,000 imperfections with the theoretical probability density function $f(\zeta)$ of normalized critical load increment ζ given in Theorem 5. The theoretical curve is in good agreement with the observed histogram. The sample mean $E[\zeta] = 0.6097$ and the sample variance $\mathrm{Var}[\zeta] = 0.5205$ observed in this experiment are very close to the theoretical values $E[\zeta] = 0.6102$ and $\mathrm{Var}[\zeta] = 0.5207$ given in Theorem 5. This assesses the validity of the present method.

3.2. *Limit point and unstable-symmetric point of bifurcation*

The shape of probability density function has strong dependency on the number p of imperfection parameters, as we have already seen in Fig. 2. In order to produce numerical examples with various values of p, we consider the elastic n-bar trusses ($n = 3, 5$) shown in Fig. 6, the elastic triangular truss dome shown in Fig. 7, and the elastic regular-hexagonal truss dome shown in Fig. 8. Further, for each of the n-bar trusses of Fig. 6 the height h of the top node is chosen to be either 0.5 or 3; for the triangular dome of Fig. 7 the height h_1 of the lower story and the height h_2 of the top node are chosen to be either $(h_1, h_2) = (10, 20)$ or (20, 40). Two types of vertical parametric loadings are applied to the regular-hexagonal dome of Fig. 8. These loadings include (a) uniform vertical loads λ applied to each free node, and (b) vertical loads applied in the proportion of 0.5λ for the crown node and λ for other free nodes. All the other structures are subjected to the loading (a).

A finite displacement analysis of these trusses has been performed to reveal that their load-bearing capacities are governed by either a limit point or an unstable-symmetric point of bifurcation. Table 2 summarizes the height, the loading, and the structure employed for each type of critical point.

All members of these structures have the same modulus of elasticity E and the same cross-section A for the perfect case. We choose the cross-section A_m of the mth member ($m = 1, \ldots, p$) as imperfection parameters, and define an imperfection parameter vector

$$\mathbf{v} = (A_1, \ldots, A_p)^T,$$

which is equal to

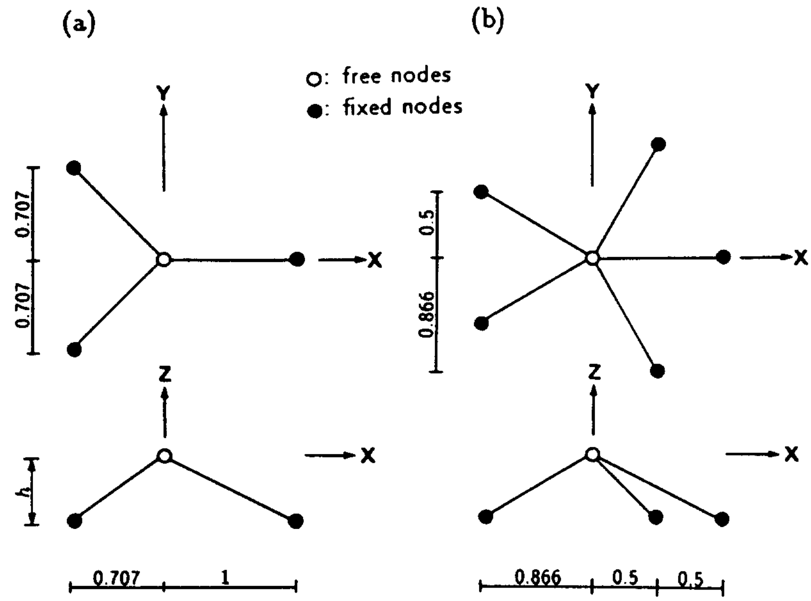

Fig. 6. n-bar trusses ($p = n$). (a) $n = 3$. (b) $n = 5$.

$$\mathbf{v}^0 = (A, \ldots, A)^T$$

for the perfect structure. The dimension p of the imperfection vector $\mathbf{v}$ for the n-bar trusses is equal to n, whereas p is equal to 12 for the triangular dome and 24 for the hexagonal one. The weight matrix is selected to be the unit matrix, i.e. $W = I_p$.

The explicit form of the imperfection sensitivity matrix B for the imperfection variables A_m ($m = 1, \ldots, p$) has already been obtained in Ikeda and Murota (1990b) as follows:

$$B = [B_{km}] = \left[-E\left(\frac{1}{L_{ij}^m} - \frac{1}{\hat{L}_{ij}^m}\right)(\mathbf{u}_i - \mathbf{u}_j)(\delta_{ki} - \delta_{kj}) \right]\Bigg|_{\mathbf{u}=\mathbf{u}_c^0} \tag{21}$$

where L_{ij}^m and $\hat{L}_{ij}^m$ denote the initial and deformed member lengths of the mth member connecting the ith and jth nodes, respectively; $\mathbf{u}_i$ and $\mathbf{u}_j$ are the position vectors after

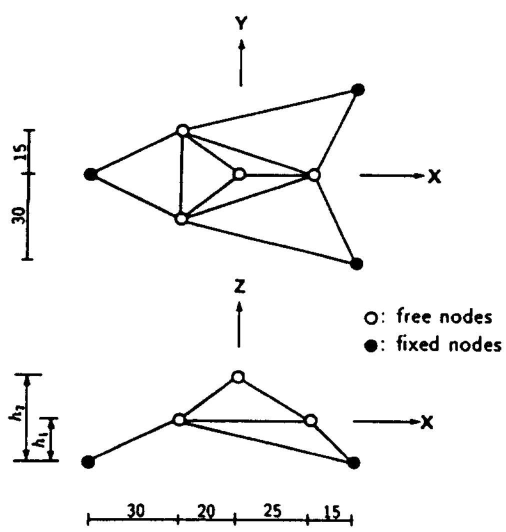

Fig. 7. Triangular truss dome structure ($p = 12$).

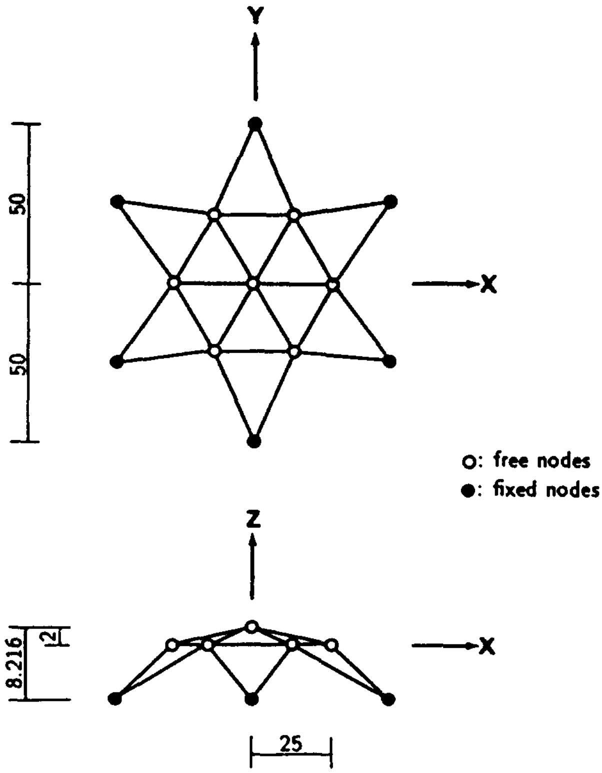

Fig. 8. Regular-hexagonal truss dome structure ($p = 24$).

displacement for the ith and jth nodes, respectively; and δ_{ij} expresses the Kronecker delta, defined to be zero or one according to whether $i \neq j$ or $i = j$.

With the use of B of (21), the critical imperfection $\mathbf{d}^*$ of (8) was computed. For a limit point the critical loads for $\mathbf{d}^*$ and $-\mathbf{d}^*$ were computed through a finite displacement analysis to obtain the lower and upper bounds of the critical load increment $\tilde{\lambda}_c$. For a symmetric point of bifurcation the critical load for $\mathbf{d}^*$ was computed to obtain its lower bound (its upper bound is equal to zero). Here and in the sequel imperfection magnitudes were chosen to be $\varepsilon = 10^{-4}$ for $p = 3, 5$ and 12; and $\varepsilon = 10^{-3}$ for $p = 24$.

For each truss structure, we have randomly chosen 100 imperfection modes $\mathbf{d}$ and traced the equilibrium paths to compute the normalized critical load increment ζ. This histogram obtained from these 100 imperfections and the theoretical probability density function $f(\zeta)$ given in Theorem 5 are compared in Fig. 9 for a limit point and in Fig. 10 for an unstable-symmetric point of bifurcation. The theoretical curves are in relatively good agreement with the observed histogram for each case. Nonetheless, 100 random

Table 2. Height, loading and structure employed for each case

p	Type of structure	Height or loading
	(a) Limit point	
3	Three-bar truss	$h = 0.5$
5	Five-bar truss	$h = 0.5$
12	Triangular truss dome	$(h_1, h_2) = (10, 20)$
24	Regular-hexagonal truss dome	Loading (a)
	(b) Unstable-symmetric point of bifurcation	
3	Three-bar truss	$h = 3$
5	Five-bar truss	$h = 3$
12	Triangular truss dome	$(h_1, h_2) = (20, 40)$
24	Regular-hexagonal truss dome	Loading (b)

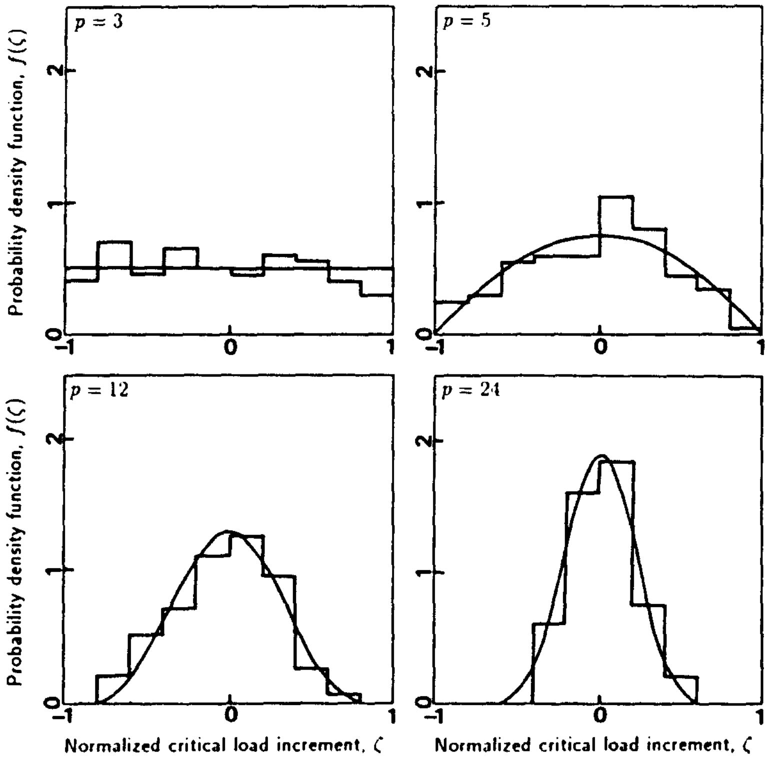

Fig. 9. Comparison of theoretical and empirical ($K = 100$) probability density functions $f(\zeta)$ of normalized critical load increment ζ for the truss structures ($p = 3, 5, 12, 24$) for a limit point. $p = 3$: three-bar truss; $p = 5$: five-bar truss; $p = 12$: triangular truss dome structure; $p = 24$: regular-hexagonal truss dome structure.

imperfections are not sufficient to yield smooth curves of empirical probability density function. This indicates the importance of the present theory.

The relationship between the average $E[\zeta]$ and the number p of imperfection variables is shown in Fig. 11, whereas Fig. 12 shows the Var $[\zeta]$ versus p relationship. The solid lines denote the theoretical values; the broken ones the asymptotic formula given in Theorem 5; the circles (●) and the diamonds (◇) represent the empirical data for a limit point and an unstable-symmetric point of bifurcation, respectively. We can see that $E[\zeta]$ and Var $[\zeta]$ tend to 0 as p increases, in agreement with the theory. In addition, the asymptotic formula is accurate enough for all p.

In the course of the analysis of random imperfections, we have obtained the minimum ζ_K for K random imperfections ($1 \leqslant K \leqslant 100$). Figure 13 shows the $|\zeta_K - \zeta_{min}|$ versus K relationships obtained for each truss structure, where $|\zeta_K - \zeta_{min}|$ represents the error of ζ_K as an approximation to the minimum load ζ_{min}. The speed of the reduction of the error $|\zeta_K - \zeta_{min}|$ tends to be slower as p increases, as anticipated by the asymptotic theory explained in Subsection 2.3. In particular, for $p = 24$ the error exceeds 37% even with $K = 100$. This shows the inefficiency of a conventional method in approximating the minimum load-bearing capacity by the minimum load achieved by a series of random imperfections.

4. CONCLUDING REMARKS

In this paper we have made a theoretical study on random initial imperfections of structures. The basic idea is to decompose the space of imperfection vectors into two orthogonal subspaces: the subspace that affects the load-bearing capacity and the other that does not. Then simple calculations have led to various kinds of stochastic properties of load-bearing capacity. The present method seems to be promising in that it permits the

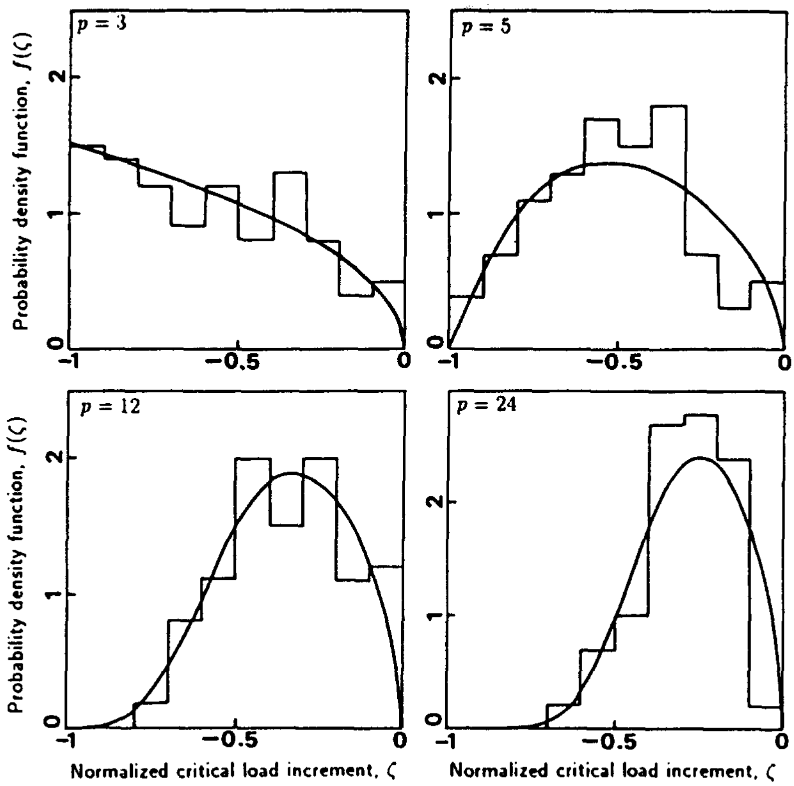

Fig. 10. Comparison of theoretical and empirical ($K = 100$) probability density functions $f(\zeta)$ of normalized critical load increment ζ for the truss structures ($p = 3$, 5, 12, 24) for an unstable-symmetric point of bifurcation. $p = 3$: three-bar truss; $p = 5$: five-bar truss; $p = 12$: triangular truss dome structure; $p = 24$: regular-hexagonal truss dome structure.

estimation of the probability density function of load-bearing capacity with much smaller cost than by the conventional method of random imperfections.

In order to apply the present analysis to structures other than trusses, one needs to derive the imperfection sensitivity matrix B for each structure. Nevertheless, the task involved in its derivation is to differentiate the equilibrium equations with respect to imperfection variables, and will be comparable to that for deriving the tangent stiffness matrix J.

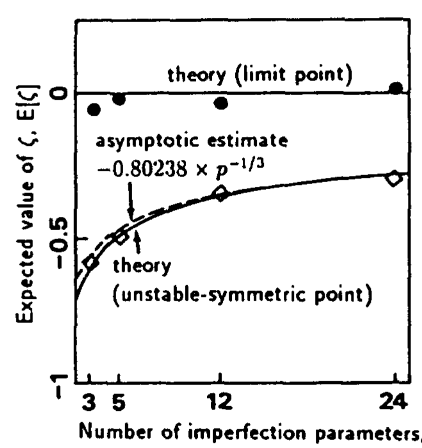

Fig. 11. Comparison of theoretical and empirical ($K = 100$) expectation $E[\zeta]$ versus p relationship. (●) Observed data for a limit point. (◇) Observed data for an unstable-symmetric point of bifurcation.

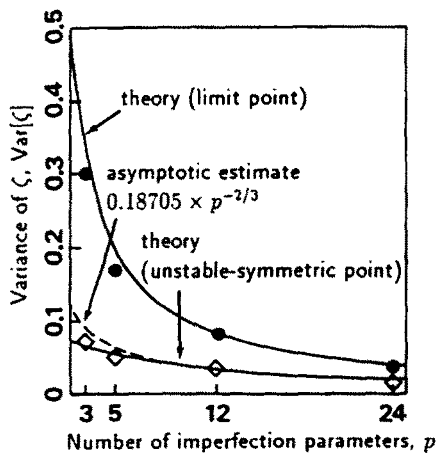

Fig. 12. Comparison of theoretical and empirical ($K = 100$) variance Var [ζ] versus p relationship. (●) Observed data for a limit point. (◇) Observed data for an unstable-symmetric point of bifurcation.

Identifying an appropriate set of imperfection parameters demands a sound engineering judgement. An introduction of redundant parameters which belong only to $\mathbf{d}_{ker}$ will not alter the minimum load but increase the number p of imperfection parameters, and in turn will sharpen the peak of the probability density function.

Though we have restricted ourselves to simple critical points in this paper, we did encounter double critical points for structures with geometric symmetry. The present approach can be extended to analyze the imperfections at double points if it is combined

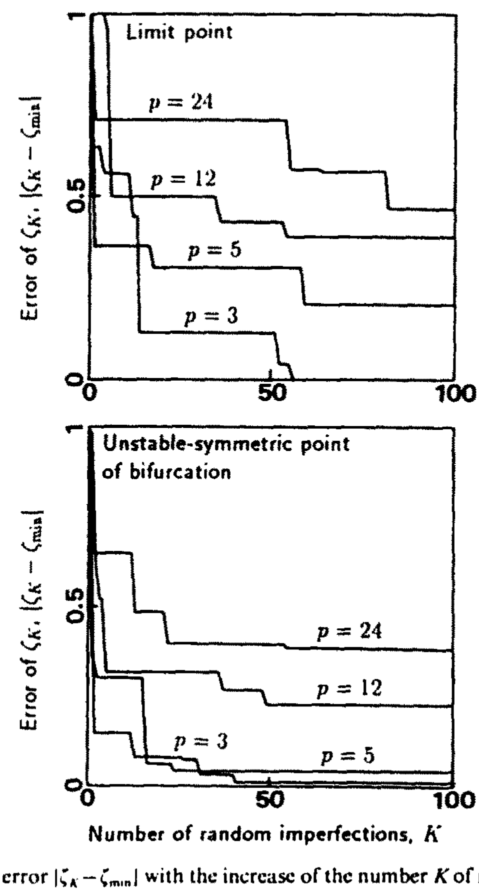

Fig. 13. The decrease of error $|\zeta_K - \zeta_{min}|$ with the increase of the number K of random imperfections.

with the group-theoretic study of critical imperfections of Murota and Ikeda (1991). This will be reported in a forthcoming paper (Murota and Ikeda, 1990).

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